Conformal expanding repellers.

September 29, 2017 2:51 PM

Now, let me define the main geometric object we work with.

Def Let $V\subset \mathbb{C}$ - open, $J\subset V$ - compact. $f: V\to\mathbb{C}$ - holomorphic. (J,V,f) is called a conformal expanding repeller (CER) if

1) $\exists C>0, \alpha>1$: $|(f^n)'(z)|\ge C\alpha^n$, $z\in J$, $n\ge 1$.
2) $\overline{f^{-1}(V)}\subset V$, $f^{-1}(V)$ - bounded, $J=\bigcap_{n\ge 1} f^{-n}(V)$. In other words, $J=\{x: f^{\circ n}(x)\in V\ \forall n\}$
3) Let U be open, $U\cap J\ne\emptyset \Rightarrow \exists n: J\subset f^n(U\cap J)$. (topological mixing).

Some easy properties:

1) $f(J)=f^{-1}(J)=J$ (J is fully invariant)

Pf $f^{-1}(J) = \bigcap f^{-n-1}(V) = \bigcap f^{-n}(V) = J$.

$x\in J \iff f^{\circ n}(x)\in V\ \forall n \Rightarrow f^{n-1}(f(x))\in V\ \forall n \Rightarrow f(x)\in J$.

If $y\in J$, then $\exists n$: $f^{\circ n}(V)\supset J$ (in fact, $J\subset f(V)$, but we do not yet know it!)
Then $\exists x\in V$: $f^{\circ n}(x)=y$. $x\in J$ ($f^{-1}(J)=J$), so $x_1:=f^{n-1}(x)\in J$, as above, $y=f(x)$. So $f(J)=J$. ∎

2) Either J is a one-point set or it is a perfect set (closed, w/o isolated points).

Pf. J is closed since $J=\bigcap f^{-n}(V)$ - closed. If J has more than one point, then if $x\in J$, $y\ne x$, $\varepsilon>0$, then, by property 3) $\exists n$: $f^{\circ n}(B(x,\varepsilon))\supset J$, so $\exists z\in B(x,\varepsilon)$: $f^{\circ n}(z)=y$. Either $z\ne x$, and we are done, or $\forall y\in J, y\ne x$, $y=f^{\circ n}(x)$. Thus $y=f(x)$, $x=f(y)$. But then $f^n(B(x,\varepsilon)\cap J)=\{x\}$ or $\{y\}\not\supset J$ — contradiction! ∎

Examples:

1) Conformal Cantor set.

$U\in\mathbb{C}$ - s.c. domain, $U_1\dots U_k\subset U$, $\overline{U_j}\subset U$, $\overline{U_j}\cap\overline{U_\ell}=\emptyset$, $j\ne \ell$.
U_j - s.c., $f_j: U_j\to U$ - conformal.
$V=\bigcup U_j$, $f:V\to\mathbb{C}$ defined by $f|_{U_j}=f_j$.
$J=\bigcap f^{-n}(V)$. Consider $(J,\bigcup U_j,f)$.

To establish that it is indeed a CER, we need to use hyperbolic metric.
It is defined on the unit disk $\mathbb{D}$ by $d\rho_{\mathbb{D}}(z):=\frac{|dz|}{1-|z|^2}$ ($\rho_{\mathbb{D}}(z_1,z_2)$ - length of the shortest curve, i.e. $\inf_{\substack{\gamma(0)=z_1\\ \gamma(1)=z_2}}\int d\rho_{\mathbb{D}}(z)$). It is invariant under conformal self-maps $\mathbb{D}$ to itself. Thus, for a s.c. domain Ω, $d\rho_\Omega(z)=(f'(f^{-1}(z)))^{-1}d\rho_{\mathbb{D}}(z)$, where $f:\mathbb{D}\to\Omega$ - any conformal map, is well-defined. Notation: $\eta_\Omega(z):=\frac{d\rho_\Omega}{|dz|}$
Classical Schwarz lemma $f:\mathbb{D}\to\mathbb{D}$, $f(0)=0 \Rightarrow |f(z)|\le|z| \iff \rho_{\mathbb{D}}(0,z)\ge\rho_{\mathbb{D}}(0,f(z))$, with equality at some pt iff f is a rotation, can be, by invariance, interpreted as:

Lemma If $f:\Omega_1\to\Omega_2$ - holomorphic then
$d_{\Omega_2}(f(z_1),f(z_2))\le d_{\Omega_1}(z_1,z_2)$, with equality reached only if $f:\Omega_1\to\Omega_2$ - conformal onto.

Corollary. If $\Omega_1\subsetneq\Omega_2$, $z_1,z_2\in\Omega_1 \Rightarrow$
$\rho_{\Omega_1}(z_1,z_2)>\rho_{\Omega_2}(z_1,z_2)$. (consider the inclusion $i:\Omega_1\to\Omega_2$, which is not onto).

In particular, if $\Omega_1\Subset\Omega_2$ (compact), we can consider $\Omega_1\subset\Omega_3\subsetneq\Omega_2$, then $\frac{\rho_{\Omega_3}(z_1,z_2)}{\rho_{\Omega_2}(z_1,z_2)}>1$ on $\Omega_3\times\Omega_3$, - thus $\exists\alpha>1$: $\frac{\rho_{\Omega_1}(z_1,z_2)}{\rho_{\Omega_2}(z_1,z_2)}\ge\frac{\rho_{\Omega_3}(z_1,z_2)}{\rho_{\Omega_2}(z_1,z_2)}\ge\alpha$ on compact

$\Omega_1 \Subset \Omega_3 \Subset \Omega_2$ ... $\frac{\ldots}{\rho_{\Omega_2}(z_1,z_2)} > 1$ on $\Omega_3 \times \Omega_1$

thus $\exists\, \mathcal{L} > 1$: $\underbrace{\frac{\rho_{\Omega_1}(z_1,z_2)}{\rho_{\Omega_2}(z_1,z_2)}}_{\mathcal{L}(\Omega_1,\Omega_2)} \geq \frac{\rho_{\Omega_3}(z_1,z_2)}{\rho_{\Omega_2}(z_1,z_2)} \geq \mathcal{L}$ on compact $\overline{\Omega_1} \times \overline{\Omega_1}$

Apply all of this to our situation

We have $\rho_U(f_j^{-1}(x), f_j^{-1}(y)) \leq \frac{\rho_V(f_j^{-1}(x), f_j^{-1}(y))}{\mathcal{L}} \leq \frac{\rho_V(x,y)}{\mathcal{L}}$.

Thus, taking it n times, for any $\sigma = (j_1, \ldots, j_n)$, we have

$\rho_V(f_\sigma^{-1}(x), f_\sigma^{-1}(y)) \leq \frac{\rho_V(x,y)}{\mathcal{L}^n}$. Let now $y \to x$, to get $|(f^{\circ n})'(x)| \geq \mathcal{L}^n \frac{\eta_V(x)}{\eta_U(f^{\circ n}(x))}$. Note now, that J is compact in V, invariant, so $M^{-1} \leq \eta_U(x) \leq M$ for some M and all $x \in J$. Thus $|(f^{\circ n})'(x)| \geq M^2 \mathcal{L}^n$ $\forall x \in J$ $\forall n \geq 1$.

Also note that $\mathrm{diam}_{\rho_V}(f_\sigma^{-1}(V)) \leq \frac{\mathrm{diam}\, \rho_U(V)}{\mathcal{L}^n} \to 0$.

So if $W \cap J \neq \emptyset$, then $\exists\, x \in J \cap W$, and $B(x, \frac{\mathrm{diam}\,\rho_U(V)}{\mathcal{L}^{n-1}}) \subset W$. Then this $B(x, \frac{\mathrm{diam}\,\rho_U(V)}{\mathcal{L}^n})$ contain some $U_{\sigma} = f_\sigma^{-1} V$, so $f_\sigma(W \cap J) \supset V \supset J$.

Notice that there is now one-to-one correspondence $I : X^b \to J$, by $(x_1, \ldots, x_n) \to \bigcap_n f_{x_1}^{-1} \circ \ldots f_{x_{n+1}}^{-1}(V)$.

It conjugates T to f: $I(T(x)) = f(I(x))$.

We can also modify this setup by taking an aperiodic A, and defining:

$U_j' := U_j \setminus \bigcup_{i: A_{ji} = 0} f_i^{-1}(V)$, $V_A := \bigcup_{j=1}^{b} V_j'$, and

$J_A = \bigcap f^{-n}(V_A) \subset J$

Then $I : X_A^{b} \to J_A$, and J_A is CER.

We just need aperiodicity for 3).

2) Reflection sets.

Let $C_1, \ldots C_n$ be non-intersecting closed disks in the upper half-plane. Let r_j be the reflection with respect to ∂C_j, s be the reflexion with respect to $\mathbb{R}$.

Let now $V = \bigcup_{j=1}^{n} \mathrm{Int}(C_j) \cup \bigcup_{j=1}^{n} \mathrm{Int}(s(C_j))$.

$f_j := f|_{C_j} := s \circ r_j =: f|_{s(C_j)} =: f_{j+n}$

$f^{-1}(V) \subset V$, $|f'(z)| > 1$ on $f^{-1}(V)$.

Now f is naturally conjugate to T on X_b^A, where $A_{ij} = 1$, if ... $A_{ii} = 0$, $b = 2n$.

A is aperiodic, and $|(f^{-1})^\sigma (V)| \to 0$ as $|\sigma| \to \infty$ (the same as for Cantor set). So if $J = \bigcap f^{-n}(V)$, then (J, V, f) is CER.

3)  S^1 $J = \{|z| = 1\}$, $f(z) = z^d$, $V = \{|z| < 2\}$.

Then $|f'(z)| = d$ on J, $J = \bigcap f^{-n}(V)$.

And f is a geometric realization of T_d shift! so transitive.

Turns out that any Jordan curve J is conjugate to this one!

In all these examples, the dynamics was conjugate to the shift or sub-shift. Turns out, it is always true!

Thm. For each CER (J, V, f) $\exists$ a Markov partition, a cover of J by sets R_j, $j=1,\dots,b$, satisfying:

1). $R_j = \mathrm{Clos}(\mathrm{Int}\, R_j)$ $\forall j$

2) $\mathrm{Int}\, R_j \cap \mathrm{Int}\, R_i = \emptyset$ $i \neq j$

3) If $f(\mathrm{Int}\, R_i) \cap \mathrm{Int}\, R_j \neq \emptyset \Rightarrow R_j \subset f(R_i)$.

4) $f|_{R_i}$ is injective.

It can be chosen so that $\max(R_i)$ is arbitrarily small.

Assume the theorem. Let us define a matrix A_{ij} by

$$A_{ij} = \begin{cases} 1, & \text{if } f(\mathrm{Int}\, R_i) \cap \mathrm{Int}\, R_j \neq \emptyset \\ 0, & f(\mathrm{Int}\, R_i) \cap \mathrm{Int}\, R_j = \emptyset. \end{cases}$$

Define $\pi : X_A^b \to J$, by $\pi((x_0, x_1, \dots x_n)) = \bigcap_{n=0}^{\infty} f^{-n}(R_{x_n})$.

Then π is well-defined (because $\pi(C(x_1,\dots,x_k)) = \bigcap_{n=0}^{k} f^{-n}(R_{x_n})$ is a non-empty compact, and $|\pi(C(x_1,\dots,x_k))| \to 0$ as $k\to\infty$, since $f^{\circ k}|_{\cap f^{-n}(R_{x_n})}$ is injective, $(f^{\circ k})' \geq C\lambda^k$)

$f \circ \pi = \pi \circ T_b$, by definition. (semi-conjugate).

π is Hölder continuous, since $\rho((x_n),(x_n')) \leq 2^{-k-1} \Rightarrow$
$(x_n),(x_n') \in C(x_1,\dots,x_k) \Rightarrow \pi((x_n)), \pi((x_n')) \in \bigcap_{n=0}^{k} f^{-n}(R_{x_n})$,
and $|\bigcap_{n=0}^{k} \dots| \to 0$ exponentially fast, i.e. $\leq C\lambda^{-k}|J|$.
We'll have a more precise estimate of the decay in the next section.

π is injective on $\pi^{-1}(J \setminus \bigcup T^{-n}(\bigcup \partial R_i))$, by the definition.
π is onto, since $\pi(X_A^b)$ is compact in J, and it contains $J \setminus \bigcup T^{-n}(\bigcup \partial R_i)$ which is dense in J.

Finally, A is aperiodic, since for some n, $f^{\circ n}(\mathrm{Int}\, R_i) \supset J$, by the mixing property. Thus we proved:

Thm. Any CER is Hölder semi-conjugate to some (X_A^b, T_b) with aperiodic A.

Now let us establish the existence of Markov partition:

Step 1. f is expanding on J.

More specifically, $\exists \beta > 0 : \forall x \neq y \in J, \exists n : |f^n(x) - f^n(y)| \geq \beta$.

Pf. Choose k, so that $\frac{1}{2} C\lambda^k =: \lambda' > 1$. Then $|(f^k)'(x)| > \lambda'$ $\forall x \in J$. Pick $\varepsilon > 0 : |x-y| < \varepsilon \Rightarrow$
$|f^k(x) - f^k(y)| \geq \lambda'$ (exists by compactness of J).

Pick $\beta = \varepsilon$, and consider $f^{kn}(x), f^{kn}(y)$. The distance between the subsequent iterates grows exponentially, till it becomes at least β. ∎

Step 2. Pseudo orbits are almost orbits.

Def Let $\eta > 0$. A sequence (x_i), $x_i \in J$, is called an η-pseudo-orbit, if $|x_{i+1} - f(x_i)| \le \eta$ $\forall i \ge 1$.

Thus the usual orbit $x_{i+1} = f(x_i)$ is a 0-orbit.

Two pseudoorbits of the same length are δ-close if $|x_i - y_i| \le \delta$ $\forall i$.

Lemma. $\forall \delta > 0$ $\exists \eta > 0$: $\forall \eta$-pseudoorbit is δ-close to some orbit.

Pf.

First, notice that if $M = \max(\sup_{x \in J} |f'(x)|, 2)$ then any $\frac{\eta}{M^k}$-pseudoorbit of f is also an η-pseudo-orbit of $f^{\circ k}$. Also, if the said pseudo-orbit is $\frac{\delta}{M^k}$ close to an orbit of $f^{\circ k}$, then the pseudoorbit of f is δ-close to an orbit of f.

Thus, we can work with $\tilde f := f^{\circ k}$, such that $|\tilde f'(x)| > \lambda' > 1$ on J and $|x - y| < \varepsilon \Rightarrow |f(x) - f(y)| > \lambda' |x - y|$.

Pick now $\eta < \varepsilon$, and also $\eta < \operatorname{dist}(J, \partial V)$.

Pick $N > 0$. The set $f^{-1}(B(x_N, \eta))$ consists of finitely many topological disks of diameter at most η/λ', on which f is injective. One of them contain x_{N-1}, so we can pick $x_{N-1,N}$ with the property $f(x_{N-1,N}) = x_N$, $|x_{N-1,N} - x_{N-1}| < \frac{\eta}{\lambda'}$.

Let $x_{N,N} = x_N$, $f(x_{N-1}) \in B(x_N, \eta)$

Also $|x_{N-1,N} - f(x_{N-2})| \le |x_{N-1,N} - x_{N-1}| + |x_{N-1} - f(x_{N-2})| < (1 + (\lambda')^{-1})\eta$.

Continue to pick $x_{k,N}$ with the property $f(x_{k,N}) = x_{k+1,N}$, $|x_{k,N} - x_k| < (1 + (\lambda')^{-1} + \dots (\lambda')^{-N-k})\eta$.

so $(x_{k,N})_{k \le N}$ is δ-close to $(x_k)_{k \le N}$ for

$$\delta = \left(\sum_{k=0}^{\infty} (\lambda')^{-k}\right)\eta .$$

Now, let us select, by diagonal process, a subsequence of x_{0N}, so that for each k, a sequence x_{k,N_ℓ} converges. Let $x'_k := \lim_{\ell \to \infty} x_{k,N_\ell}$. Then $x'_{k+1} = f(x'_k)$, and $|x'_k - x_k| \le \delta$ $\forall k$ ∎ -close to

Step 3. Constructing a pre-Markov partition.

Take β from Step 1. Take any $\delta < \frac{1}{2}\beta$. Take $\gamma < \frac{1}{2}\eta$ (η is from step 2, for δ), such that $|x - y| \le \gamma \Rightarrow |f(x) - f(y)| \le \frac{1}{2}\eta$.

Let $\Sigma = \{x_1, \dots, x_n\}$ be a finite γ-net for J (i.e. $\forall x \in J$ $\exists i$: $|x_i - x| \le \gamma$).

Let $\Sigma(f) \subset \Sigma^{\mathbb{N}}$ consists of η-pseudoorbits - a sequences (y_i), $|f(y_i) - y_{i+1}| \le \eta$, $y_i = x_i$ for some i.

By Step 2, $(y_i) \in \Sigma(f) \Rightarrow \exists x \in J : \forall i$ $|y_i - f^i(x)| \le \delta$.

By Step 1, such an x is unique ($2\delta < \beta$, so if for all n $|f^n(x) - f^n(y)| \le 2\delta \Rightarrow x = y$).

Write $x = \theta((y_i))$. θ is well-defined, surjective (since

Σ forms a γ-net and $|f^k(x) - y_k| < \delta \Rightarrow |f^{k+1}(x) - f(y_k)| < \eta/2$, so if y_{k+1} is the closest element of Σ to $f^{(k+1)}(x)$, then $|y_{k+1} - f(y_k)| < \eta$.

$\Sigma(f)$ is closed, in product topology, in $\Sigma^{\mathbb{N}}$.
By Step 1, θ is continuous.
Define $F_k = \theta\{(y_i) \in \Sigma(f) : y_0 = x_k\}$.
Then $F_k \subset J$ - closed, $\bigcup F_k = J$ (since θ is surjective).
Also if $f(F_k) \cap F_i \neq \emptyset \Rightarrow \exists (y_i) \in \Sigma(f)$ with $y_0 = x_k$, $y_1 = x_i$. Thus $|f(x_k) - x_i| \leq \eta$. Thus if $x \in F_i$, $x = \theta((y_j), y_0 = x)$, then $(x_k y) \in \Sigma(f)$, and $f(\theta(x_k y)) = x$, $\theta(x_k y) \in F_k$. So $F_i \subset f(F_k)$.
Also, if $x, x' \in F_k$, they correspond to different $\theta(x_k y), \theta(x_k y')$, so $f(x) = \theta(y) \neq \theta(y') = f(x')$. So $f|_{F_k}$ is injective.
Thus, conditions 3) and 4) of Markov are satisfied.

Step 4 Constructing Markov partition

$Z := J \setminus \bigcup \partial F_i$ - open and dense subset of J (by Baire).
Define $D_{jk} := F_j \setminus F_k$, $I_{jk} := F_j \cap F_k$. $F_j = D_{jk} \cup I_{jk}$ $\forall k$.
For $x \in Z$, define $R(x) = \bigcap_{x \in \mathrm{Int}\, I_{jk}} \mathrm{Int}\, I_{jk} \cap \bigcap_{x \in \mathrm{Int}\, D_{jk}} \mathrm{Int}\, D_{jk}$ (non-empty, since $x \in Z$)
If $x, y \in Z$, $R(x) \cap R(y) \neq \emptyset \Rightarrow R(x) = R(y)$ — we have to intersect the same sets).
There are only finitely many such sets, let $(R_1, \dots, R_n)$ be the collection of their closures: $(\mathrm{Clos}\, R(x))_{x \in Z}$.
Then $R_j = \mathrm{Clos}(\mathrm{Int}\, R_j)$, $R_j \subset F_l$ for some l, so $f|_{R_j}$ is injective. $\mathrm{Int}\, R_j \cap \mathrm{Int}\, R_k = \emptyset$ if $j \neq k$, and $\bigcup R_j = J$, since Z is everywhere dense, and every point of Z belongs to some R_j.

Let us check 3). Assume that $y \in f(R(x))$ for some x. Then, since f is injective on $R(x)$, and f is open (as conformal map),
$f(R(x)) = \bigcap \mathrm{Int}(f(F_j \cap F_k)) \cap \bigcap \mathrm{Int}(f(F_j \setminus F_k)) =$
$\bigcap \mathrm{Int}(f(F_j) \cap f(F_k)) \cap \bigcap \mathrm{Int}(f(F_j) \setminus f(F_k))$.
Since each $f(F_j) = \bigcup_{k \in S_j} F_k$ for some $S_j \subseteq \{1, \dots, n\}$,
we see that $f(R(x)) = \bigcup_{y \in f(R(x))} R(y)$. So 3) is satisfied ■

Using Markov partition and the semi-conjugacy π, as above, we can talk about $P(\varphi)$, Gibbs and equilibrium measures for Hölder functions defined on J. The corresponding measures will also live on J.